CHAPTER 6

Sections 6.1 & 6.2

SECTION 6.1

Directed Graphs

Directed Graphs

- Consider an unweighted, directed graph with no parallel arcs and n nodes.
  - Adjacency matrix will be an n x n matrix.
  - The matrix will be a Boolean matrix.
  - There is a one-to-one correspondence between a directed graph with n nodes and no parallel arcs, and an n x n Boolean matrix.
    - Given one, the other can be constructed.

Binary Relations

- There is also a one-to-one correspondence between binary relations on n-element sets and directed graphs with n nodes and no parallel arcs.

- Given a directed graph,
  - If there is an arc from node n_i to node n_j, then (n_i, n_j) is a member of the corresponding **adjacency relation**.

- Example 1
  - Consider the directed graph of Figure 6.1.
  - What is the adjacency relation that corresponds to the graph?

Example 2

- Example 2
  - For the set $N = \{1, 2, 3, 4\}$ and the binary relation $\{(1,4), (2,3), (2,4), (4,1)\}$ on N
  - Draw the associated directed graph.

Binary relations

- All 3 sets are equivalent.
  - Binary relations on n-element sets
  - Directed graphs with n nodes and no parallel arcs
  - n x n Boolean matrices
- An item from any of the 3 sets has corresponding representations in the other 2 sets.

Reachability

- It is often useful to be able to test a node for reachability.
- Examples
  - Data flow diagram: if a node is unreachable, it cannot affect the software system, and can be eliminated.
  - In a transportation system (such as roads or airline routes), you don't want a node (an airport or city) to be unreachable.

Reachability

- Recall the definition for a reachable node
  - In a directed graph, node n_j is reachable from node n_i if there is a path from n_i to n_j.
- Consider the Boolean matrix for the graph in Figure 6.1
  - 1 in position i,j indicates a path of length 1 exists from node n_i to node n_j.
  - Now calculate $\mathbf{A}^{(2)} = \mathbf{A} \times \mathbf{A}$

Reachability Matrix

- For an n x n matrix A
  - The **reachability matrix** $\mathbf{R}$ is the Boolean sum of $\mathbf{A}, \mathbf{A}^{(2)}, \mathbf{A}^{(3)}, \ldots, \mathbf{A}^{(n)}$
- Example 5
  - Let G be the directed graph in Figure 6.3
  - What is the adjacency matrix $\mathbf{A}$ for G?
  - What is the adjacency relation ρ for G?
  - What is the reachability matrix $\mathbf{R}$ for G?

Reachability

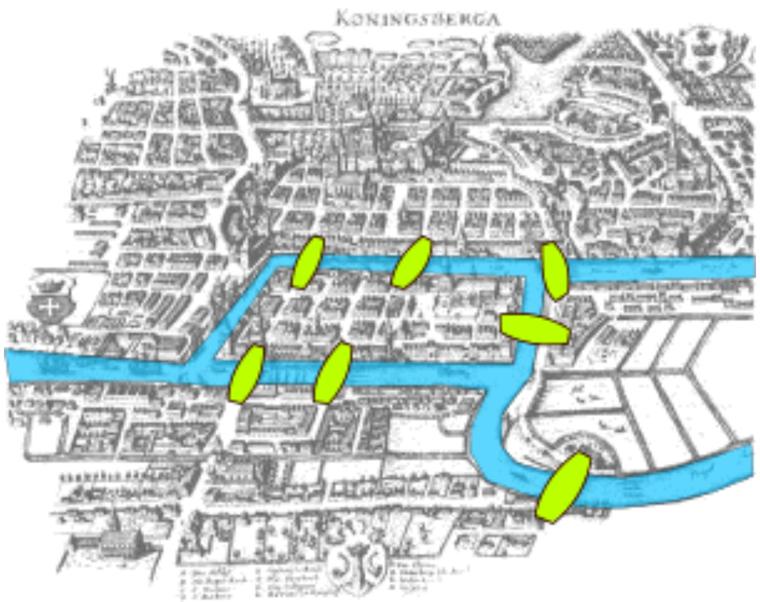
- The work done in calculating **R** is order n^4.
- Warshall's Algorithm
  - A more efficient method for computing the reachability matrix of a graph (or the transitive closure of a relation)
  - Warshall's Algorithm is order n^3

SECTION 6.2

Euler Path and Hamiltonian Circuit

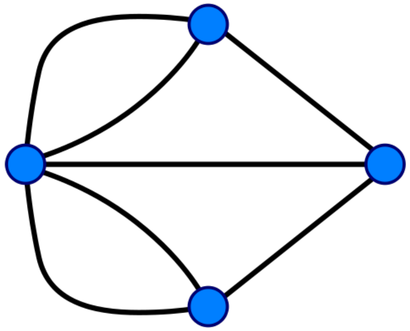
Euler Path Problem

- Bridges of Königsberg (18th century)
  - Can you walk through the city crossing each bridge only once?



Euler Path Problem

- Euler solved the general problem mathematically, laying the foundations of graph theory
- More general problem
  - Does an **Euler path** exist in a given graph?
  - **Euler path** in a graph G is a path that uses each arc of G exactly once.



Existence of an Euler Path

- Depends on degree of its nodes
  - A node is even if its degree is even
  - A node is odd is its degree is odd
  - Theorem: The number of odd nodes in any graph is even (Any graph has an even number of nodes with odd degree).
- Euler Path exists in a connected graph iff:
  - either there are no odd nodes
    - Path can begin at any node and will end there
  - or there are 2 odd nodes
    - Path must begin at one odd node and end at the other

Euler Path

- Practice 7
  - Work using the Theorem on Euler Paths.

- Euler Path algorithm is listed in book on page 494.
  - Uses adjacency matrix

Hamiltonian Circuit

- Hamiltonian Circuit
  - A cycle using every node of the graph exactly once
    - Except the start node, which is also the end node
  - No arc can be used more than once
    - An arc does not have to be used
- No efficient algorithm has ever been found to determine if a Hamiltonian circuit exists