CHAPTER 4

Section 4.6

Matrices

- Matrix
  - Represents values in rows and columns

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 3 & -6 & 8 \end{bmatrix}$$

  - The ***dimensions*** of a matrix are the number of rows and columns.
    - **A** is a 2×3 matrix.
  - Elements of **A** are denoted by a_{ij}, where
    - i is the row number and
    - j is the column number

    of the element in the matrix.

Practice 50

- In the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -6 & 8 \\ 3 & 0 & 1 & -7 \end{bmatrix}$$

- What is a_{23}? What is a_{24}? What is a_{13}?

Matrices

- For 2 matrices to be equal,
  - they must have the same dimensions and the same entries in each location.
- Example 63
  - Let

$$\mathbf{X} = \begin{bmatrix} x & 4 \\ 1 & y \\ z & 0 \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \\ 2 & w \end{bmatrix}$$

  - If $\mathbf{X} = \mathbf{Y}$, then $x = 3$, $y = 6$, $z = 2$ and $w = 0$.

Square Matrices

- In a square matrix
  - The number of row equals the number of columns.

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 7 \\ 5 & 0 & 2 \\ 7 & 2 & 6 \end{bmatrix}$$

  - **A** is a 3×3 matrix. The elements a_{11}, a_{22}, and a_{33} form the main diagonal of the matrix.
  - In general, if A is an $n \times n$ matrix, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ form the main diagonal of the matrix.

Matrices and Arrays

- A matrix is a 2-dimensional array.
  - To represent a matrix in a programming language, you can use a 2-dimensional array.
- A 1-dimensional array is called a **vector**.

Matrix Operations

- Scalar multiplication
  - Multiply each entry of a matrix by a fixed single number called a **scalar**.
  - Example 65: Multiply **A** by the scalar $r = 3$.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 6 & -3 & 2 \end{bmatrix}$$

  - What is $r\mathbf{A}$?

Matrix Addition

- To add 2 matrices
  - The matrices must have the same dimensions
  - Add the corresponding elements.
  - Example 66

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 0 & 4 \\ -4 & 5 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & -2 & 8 \\ 1 & 5 & 2 \\ 2 & 3 & 3 \end{bmatrix}$$

  - What is $\mathbf{A} + \mathbf{B}$?

Practice 51

- For $r = 2$

$$\mathbf{A} = \begin{bmatrix} 1 & 7 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 9 & 2 \\ -1 & 4 \end{bmatrix}$$

- Find $r\mathbf{A} + \mathbf{B}$.

Matrix Subtraction

- Subtraction: $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} 1 & 7 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 9 & 2 \\ -1 & 4 \end{bmatrix}$$

Zero Matrix

- Zero matrix
  - All entries are 0

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- What happens if we add $\mathbf{0}$ to some matrix $\mathbf{A}$?

Example 67

- If $\mathbf{A}$ and $\mathbf{B}$ are $n \times m$ matrices and r and s are scalars
  - the following matrix equations are true.

$$\mathbf{0} + \mathbf{A} = \mathbf{A}$$

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

$$r(A + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$$

$$(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$$

$$r(s\mathbf{A}) = (rs)\mathbf{A}$$

Matrix Multiplication

- To compute $\mathbf{A} \cdot \mathbf{B}$,
  - The number of columns in $\mathbf{A}$ must match the number of rows in $\mathbf{B}$
    - If $\mathbf{A}$ is an n x m matrix, then $\mathbf{B}$ must be an m x p matrix
    - The result is an n x p matrix
  - Let $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$. Then

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

  - An entry in row i, column j of $\mathbf{C}$ is calculated by multiplying elements in row i of $\mathbf{A}$ by the corresponding elements in column j of $\mathbf{B}$ and adding the results.

Matrix Multiplication

- Example 68
  - Find $\mathbf{A} \cdot \mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ 4 & -1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 3 \\ 2 & 2 \\ 6 & 5 \end{bmatrix}$$

Matrix Multiplication

- Practice 52
  - Compute $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 6 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix}$$

Matrices Equations

- Example 69
  - Given matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ of appropriate dimensions, and
  - the scalar values r and s, then

$$\mathbf{A}(\mathbf{B}\cdot\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$

$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{A}\cdot\mathbf{C}+\mathbf{B}\cdot\mathbf{C}$$

$$r\mathbf{A}\cdot s\mathbf{B} = (rs)(\mathbf{A}\cdot\mathbf{B})$$

Identity Matrix

- **Identity matrix**: an n x n matrix with 1s along the main diagonal and 0s elsewhere
  - The following is the identity matrix with n = 4.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity Matrix

- The following equation is true for $\mathbf{I}$, where $\mathbf{A}$ is an n x n matrix.

$$\mathbf{I} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{I} = \mathbf{A}$$

  - Practice 53: Verify that $\mathbf{I} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{I} = \mathbf{A}$ given the following $\mathbf{A}$ and $\mathbf{I}$.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Identity Matrix

- An n x n matrix **A** is invertible if there exists an n x n matrix **B** such that

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = \mathbf{I}$$

  - **B** is the **inverse** of **A**, denoted by $\mathbf{A}^{-1}$.
  - Ex. 70: Given **A** and **B** below, show $\mathbf{B} = \mathbf{A}^{-1}$

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

Boolean Matrices

- A **Boolean matrix** consists of only 0s and 1s.
- Boolean operations
  - **Boolean addition**: $x \vee y = \max(x, y)$
  - To compute **A** ∨ **B**, combine corresponding elements using Boolean addition

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Boolean Matrices

- Boolean operations
  - **Boolean multiplication**: $x \wedge y = \min(x, y)$
  - To compute $\mathbf{A} \wedge \mathbf{B}$, combine corresponding elements using Boolean multiplication
- Compute $\mathbf{A} \wedge \mathbf{B}$ for the previous matrices.

Boolean Matrix Multiplication

- **Boolean matrix multiplication** $(\mathbf{A} \times \mathbf{B})$ can be defined in terms of Boolean addition and Boolean multiplication

$$c_{ij} = \bigvee_{k=1}^{m} (a_{ij} \wedge b_{kj})$$

- Compute $\mathbf{A} \times \mathbf{B}$ on the previous matrices.
- Practice 56: Does $\mathbf{A} \times \mathbf{B} = \mathbf{A} \cdot \mathbf{B}$?
- Practice 57: Compute $\mathbf{B} \times \mathbf{A}$.