CHAPTER 4

Section 4.4

Functions

- Special type of binary relations
- Three parts to a function
  - The set of starting values (Domain)
  - The set from which associated values come from (Codomain)
  - The association itself.
- Figure 4.15, page 333

Terminology for Functions

- Let S and T be sets.
  - A **function** (**mapping**) f from S to T, $f:S \rightarrow T$, is a subset of $S \times T$ where each member of S appears exactly once as the first component of an ordered pair.
    - Functions cannot be one-to-many or many-to-many
  - S is the **domain** and T is the **codomain** of the function
  - If (s,t) belongs to f, then $t = f(s)$; t is the **image** of s under f, s is a **preimage** of t under f, and f is said to map s to t
  - For $A \subseteq S$, $f(A)$ denotes $\{f(a) \mid a \in A\}$.

Practice 23

Which of the following are functions from the domain to the codomain indicated?

a. $f: S \to T$ where $S = T = \{1, 2, 3\}$,
   $f = \{(1,1), (2,3), (3,1), (2,1)\}$

b. g: Z → N where g is defined by
   $g(x) = |x|$ (the absolute value of x)

c. h:N → N where h is defined by
   $h(x) = x - 4$

Equal Functions

- Equal functions
  - Two functions are equal if they have all of the following
    - The same domain
    - The same codomain
    - The same association of values of the codomain with values of the domain
  - To show two functions with the same domain and same codomain are equal, you must show the associations are the same (they map given values to the same place).

Practice 27

- Let $S = \{1, 2, 3\}$ and $T = \{1, 4, 9\}$. The function $f: S \rightarrow T$ is defined by $f = \{(1,1), (2,4), (3,9)\}$. The function $g: S \rightarrow T$ is defined by the equation

$$g(n) = \frac{\sum_{k=1}^{n} (4k - 2)}{2}$$

- Prove that $f = g$.

Onto Functions

- Let f: $S \rightarrow T$ be an arbitrary function with domain S and codomain T.
  - Every member of S has an image under f and all the images are members of T;
  - The set R of all such images is called the range of the function f
    - $R = \{f(s) \mid s \in S\}$, or $R = f(S)$
    - R is a subset of T, $R \subseteq T$
  - If $R = T$, then the function is an **onto,** or **surjective,** function

One-to-One Functions

- A function $f: S \rightarrow T$ is **one-to-one**, or **injective**, if no member of T is the image under f of two distinct elements of S.
- To prove a function is not one-to-one, give a counter example, an element in the range with 2 preimages in the domain.

Examples

- Figure 4.20 in the text.
- Determine if they are functions, one-to-one, and/or onto.

Bijective Functions

- A function $f: S \rightarrow T$ is **bijective** (a **bijection**) if it is both one-to-one and onto.
  - Is f: R → R given by $f(x) = x^3$ a bijection?
  - Is f: R → R given by $f(x) = x^2$ a bijection?

Composition Function

- Let f and g be functions with f: $S \rightarrow T$ and g: $T \rightarrow U$.
  - For any $s \in S$, $f(s)$ is a member of T, but T is the domain of g.
  - The function g can be applied to $f(s)$, which yields $g(f(s))$, a member of U.
  - Essentially, we've mapped s to a member of U (i.e., we now have a function from S to U).
  - The function is called the **composition** function and is denoted as $(g \circ f)(s) = g(f(s))$

Composition Function

- Definition: Let $f: S \to T$ and $g: T \to U$. Then the composition function, $g \circ f$, is a function from S to U defined by $(g \circ f)(s) = g(f(s))$.
- $g \circ f$ is applied right to left
  - f is applied first, then g.
- What if $f: S \to T$ and $g: W \to Z$, where T and W are disjoint? What is $g \circ f$?

Composition Function

- Practice 31
  - Let f: R → R be defined by $f(x) = x^2$.
  - Let g: R → R be defined by $g(x) = \lfloor x \rfloor$.
    - What is the value of $(g \circ f)(2.3)$?
    - What is the value of $(f \circ g)(2.3)$?

Composition Function

- Composition function preserves the onto and one-to-one properties.
  - If both f and g are onto functions, then $g \circ f$ is an onto function.
  - If both f and g are one-to-one functions, then $g \circ f$ is one-to-one.
- What does this tell you about the composition of two bijections?

Inverse Functions

- Let $f: S \to T$ be a bijection.
  - f is onto, so every $t \in T$ has a preimage in S.
  - f is one-to-one, so the preimage is unique.
  - We can map each element t of T to a unique element of S, such that $f(s) = t$.
    - This is a function g from T to S, $g: T \to S$, and $g(t) = s$
  - The composition of f and g, $g \circ f$ maps every element in S back to itself.
    - $(g \circ f)(s) = g(f(s)) = g(t) = s$
    - The function that maps each element in a set S to itself is called the **identity** function on S and is denoted by i_S.
      $(g \circ f) = i_S$ and $(f \circ g)=i_T$.

Inverse Function

- Let f be a function, $f: S \to T$
  - If there exists a function $g: T \to S$ such that $g \circ f = i_S$ and $f \circ g = i_T$, then g is called the **inverse** function of f, denoted by f^{-1}.
  - There is only one inverse function for a given function.
- Theorem: Let $f: S \to T$
  - Then, f is a bijection if and only if f^{-1} exists.

Permutation Functions

- Permutations of a set
  - For a given set A,
    $S_A = \{f \mid f: A \to A \text{ and } f \text{ is a bijection}\}$
  - So, S_A is the set of all bijections of set A to itself; such functions are called **permutations** of A.
  - Permutation functions represent ordered arrangements of the objects in the domain.

Permutation Functions

- Example
  - If $A=\{1,2,3,4\}$, one permutation function f of A is given by
    $$f = \{(1, 2), (2, 3), (3, 1), (4, 4)\}$$
  - The function f can also be written in array form as follows
    $$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

Permutation Functions

- Cycle notation
  - Given the permutation function from before

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

  - The function can be written in **cycle notation** as $f = (1, 2, 3)$, which means that f maps
    - each element listed to the one on its right
    - the last element to the first, and
    - an element of the domain not listed to itself.

Permutation Functions

- Practice 35
  - Let $A = \{1, 2, 3, 4, 5\}$, and let $f \in S_A$ be given in array form by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix}$$

    - Write f in cycle form.
  - Let $A = \{1, 2, 3, 4, 5\}$ and let $g \in S_A$ be given in cycle form by $g = (2, 4, 5, 3)$. Write g in array form.

Permutation Functions

- Let A = {1, 2, 3, 4, 5}
- If f and g are members of S_A, and the cycles for f and g have no elements in common, f and g are **disjoint cycles**.
  - $f = (5, 2, 3)$ and $g = (3, 4, 1)$
  - Find $g \circ f$. Find $f \circ g$.
  - Write both in cycle form.

Identity Permutation

- The **identity permutation** is an identity function on A, i_A, that maps each element of A to itself.
- Example: $A = \{1, 2, 3, 4\}$
  - $f \in S_A$, given by $f = (1, 2)$.
  - Compute $f \circ f$.

Permutation Functions

- Some permutations of A cannot be written as a cycle.
  - Example: $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$
  - This cannot be written as a cycle, but can be written as a composition of 2 disjoint cycles.
    - Every permutation on a finite set that is not the identity permutation can be written as a composition of one or more disjoint cycles.

Derangement

- A **derangement** is a permutation that maps ***no*** element to itself.
  - Example: $A = \{1, 2, 3, 4, 5\}$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$$

  - If it can be written as a cycle, it must have each element from the set in the cycle.
    - Example: Given A from above, is $f = (3, 4, 1, 5)$ a derangement?