CHAPTER 4

Section 4.1

Relations

- Binary relation
  - A relationship between pairs of elements within a set
  - Example 1
    - Cartesian product, $S \times S$ or S^2
      - S = {1, 2, 3}
      - S^2 = {(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)}
    - Consider the **equality** relationship.
      - Which pairs of S^2 satisfy this relationship?
      - Given ordered pairs (x,y), we want those where $x=y$.

Relations

- Generally
  - $x \, \rho \, y$: the ordered pair (x,y) satisfies relation ρ
  - To define a relation ρ
    - describe it in words, or
    - list the ordered pairs that satisfy the relation.
- Example 2
  - $S = \{1,2,4\}$
  - ρ is defined by $x \, \rho \, y$ if and only if $x = y/2$
    - $x \, \rho \, y \leftrightarrow x = y/2$

Binary Relation on a Set

- Binary relation on a Set, S
  - Given a set S, a binary relation on a set S is a subset of $S \times S$ (a set of ordered pairs of elements of S).
  - $x \rho y \leftrightarrow (x,y) \in \rho$
- Typically, to define a relation, we describe it instead of listing its ordered pairs.
  - Binary relation implies a yes/no result
    - An ordered pair either does or does not belong to the relation

Relation Recap

- Binary Relation ρ on a set S
  - Defines a relationship between members of S.
    - If our relation ρ is "greater than", and 3 ρ 1 (3 is related to 1), that means 3 is greater than 1.
  - If 2 elements of S are related by ρ, they are listed as an ordered pair in ρ.
    - If x is related to y, then $(x,y) \in \rho$
    - $(x,y) \in \rho$ means x is related to y.
    - If our relation is "greater than" and 3 ρ 1, then (3,1) will be a member of the set ρ
    - Note that "x is related to y" means (x,y) is in the relation, **not** that (y,x) is in the relation. "x is related to y" is **different** from "y is related to x".

Binary Relation on a Set

- Example 3
  - Let $S = \{1,2\}$.
  - $S \times S$ = ?
  - ρ on S is defined by $x\ \rho\ y \leftrightarrow x + y$ is odd.
  - What are the members of ρ?

Relation on Multiple Sets

- Relations on multiple sets
  - Given two sets S and T, a binary relation from S to T is a subset of $S \times T$
  - Given n sets $S_1, S_2, \ldots, S_n$, $n > 2$, an n-ary relation on $S_1 \times S_2 \times \ldots \times S_\mathrm{n}$ is a subset of $S_1 \times S_2 \times \ldots \times S_\mathrm{n}$

Relations on Multiple Sets

- Example 5
  - Let $S = \{1, 2, 3\}$ and $T = \{2, 4, 7\}$
  - What is $S \times T$?
  - The set $\{(1,2), (2,4), (2,7)\}$ consists of elements from $S \times T$
    - So, the set is a binary relation from S to T
- We will be focusing on relations on a single set.

Practice 1

- For each of the following binary relations ρ on *N*, decide which of the given ordered pairs belong to ρ.
  - $x \rho y \Leftrightarrow x = y + 1$; (2,2), (2,3), (3,3), (3,2)
  - $x \rho y \Leftrightarrow x$ divides y; (2,4), (2,5), (2,6)
  - $x \rho y \Leftrightarrow x$ is odd; (2,3), (3,4), (4,5), (5,6)
  - $x \rho y \Leftrightarrow x > y^2$; (1,2), (2,1), (5,2), (6,4), (4,3)

Types of Relations

- Let ρ be a binary relation on S
  - ρ consists of a set of ordered pairs of the form (s_1, s_2)
  - There are various ways to pair s_1 and s_2
    - **One-to-one** relation: each s_1 and s_2 only appear once
    - **One-to-many** relation: some s_1 appears more than once
    - **Many-to-one** relation: some s_2 appears more than once
    - **Many-to-many** relation: at least one s_1 is paired with more than one s_2, and at least one s_2 is paired with more than one s_1

Practice 2

- $S = \{2, 5, 7, 9\}$
- Identify the following relations on S as one-to-one, one-to-many, many-to-one, or many-to-many
  - $\{ (5,2), (7,5), (9,2) \}$
  - $\{ (2,5), (5,7), (7,2) \}$
  - $\{ (7,9), (2,5), (9,9), (2,7) \}$

Binary Relations

- Suppose B is the set of all binary relations on a given set S
  - If ρ and σ belong to B, then they are subsets of $S \times S$
  - Since ρ and σ are sets, we can perform set operations on them, such as union, intersection and complementation
    - Results in new subsets of $S \times S$, which are new binary relations.
    - $\rho \cup \sigma$, $\rho \cap \sigma$, and ρ'

Binary Relations

- $\rho \cup \sigma$

  $x\ (\rho \cup \sigma)\ y \leftrightarrow x\ \rho\ y$ or $x\ \sigma\ y$

  i.e., the pair (x,y) satisfies the relation $\rho \cup \sigma$ if and only if (x,y) satisfies the relation ρ or (x,y) satisfies the relation σ

- $\rho \cap \sigma$

  $x\ (\rho \cap \sigma)\ y \leftrightarrow x\ \rho\ y$ and $x\ \sigma\ y$

- ρ'

  $x\ \rho'\ y \leftrightarrow$ not $x\ \rho\ y$

Practice 3

- Let ρ and σ be two binary relations on N, defined by
  - $x \ \rho \ y \leftrightarrow x = y$ and $x \ \sigma \ y \leftrightarrow x < y$
- Give verbal descriptions for (a), (b), and (c); give a set description for (d).
  - What is the relation $\rho \cup \sigma$?
  - What is the relation ρ'?
  - What is the relation σ'?
  - What is the relation $\rho \cap \sigma$?

Relation Facts / Identities

1a. $\rho \cup \sigma = \sigma \cup \rho$	1b. $\rho \cap \sigma = \sigma \cap \rho$
2a. $(\rho \cup \sigma) \cup \gamma$ $= \rho \cup (\sigma \cup \gamma)$	2b. $(\rho \cap \sigma) \cap \gamma$ $= \rho \cap (\sigma \cap \gamma)$
3a. $\rho \cup (\sigma \cap \gamma)$ $= (\rho \cup \sigma) \cap (\rho \cup \gamma)$	3b. $\rho \cap (\sigma \cup \gamma)$ $= (\rho \cap \sigma) \cup (\rho \cap \gamma)$
4a. $\rho \cup \varnothing = \rho$	4b. $\rho \cap S^2 = \rho$
5a. $\rho \cup \rho' = S^2$	5b. $\rho \cap \rho' = \varnothing$

Properties of Relations

- Properties of relations
  - *Reflexive*: for every element e in the set S, the ordered pair (e,e) must be a member of the relation
    - $(\forall e)(e \in S \rightarrow (e,e) \in \rho)$
    - Given $S=\{1,b\}$, the set $\{(1,1),(1,b),(b,b)\}$ is reflexive, but the set $\{(1,1),(1,b)\}$ is not reflexive.
  - *Symmetric:* If the ordered pair (x_1, x_2) is a member of the relation, then (x_2, x_1) must also be a member of the relation.
    - $(\forall x_1)(\forall x_2)(x_1 \in \mathrm{S} \wedge x_2 \in \mathrm{S} \wedge (x_1, x_2) \in \rho \rightarrow (x_2, x_1) \in \rho)$
    - Given $\mathrm{S} = \{1, 2, 3\}$
    - {(1,1),(2,1),(1,2)} is symmetric; {(1,1),(2,1)} is not.

Properties of Relations

- Properties of relations
  - *Not Symmetric:* At least one ordered pair (x_1, x_2) is in the relation where (x_2, x_1) is not in the relation.
    - {(1,1), (1,3), (3,1), (2,1)} is not symmetric because of (2,1)
  - *Antisymmetric:* there are no symmetric members of the relation where $x \neq y$.
    - $(1,1)$ can be in an antisymmetric relation
    - $(2,1)$ and $(1,2)$ cannot be together in an antisymmetric relation
    - $\{(1,1), (2,2), (3,2)\}$ is antisymmetric
    - $\{(1,1), (2,2), (3,2), (2,1), (1,2)\}$ is **not** antisymmetric.
    - Antisymmetric $\neq$ not symmetric

Properties of Relations

- More Properties
  - *Transitive:* If you have the ordered pairs (x_1, x_2) and (x_2, x_3) in the relation, then the ordered pair (x_1, x_3) must also be in the relation
    - $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 \in S \wedge x_2 \in S \wedge x_3 \in S \wedge (x_1, x_2) \in \rho \wedge (x_2, z) \in \rho \rightarrow (x_1, x_3) \in \rho)$
    - If x_1 is related to x_2, and x_2 is related to x_3, then x_1 must be related to x_3.
    - The set {(1,2),(2,1),(1,1),(2,2)} is transitive.
    - The set {(1,1),(3,1),(2,3)} is not transitive.

Example 6

- Consider the relation ≤ on the set N.
- Is the relation
  - Reflexive?
  - Transitive?
  - Symmetric?

Problem 10

- Let $S = \{0,1,2,4,6\}$.
- Are the following reflexive, symmetric, antisymmetric, and/or transitive?

a. $\rho = \{(0,0),(1,1),(2,2),(4,4),(6,6),(0,1),(1,2),(2,4),(4,6)\}$
b. $\rho = \{(0,1),(1,0),(2,4),(4,2),(4,6),(6,4)\}$
c. $\rho = \{(0,1),(1,2),(0,2),(2,0),(2,1),(1,0),(0,0),(1,1),(2,2)\}$
d. $\rho = \{(0,0),(1,1),(2,2),(4,4),(6,6),(4,6),(6,4)\}$

Closure of a Relation

- Let ρ be a relation on set S
- A binary relation ρ^* on set S is the closure of ρ with respect to property P if
  - ρ^* has property P
  - $\rho \subseteq \rho^*$
  - ρ^* is a subset of any other relation on S that includes ρ and has property P.
- What if ρ already has property P?

Practice / Problems

- Practice 6
  - Does it make sense to look for the antisymmetric closure of a relation on a set?
- Find the reflexive, symmetric, and transitive closure of the sets from Problem 10.

a. $\rho = \{(0,0),(1,1),(2,2),(4,4),(6,6),(0,1),(1,2),(2,4),(4,6)\}$

b. $\rho = \{(0,1),(1,0),(2,4),(4,2),(4,6),(6,4)\}$

c. $\rho = \{(0,1),(1,2),(0,2),(2,0),(2,1),(1,0),(0,0),(1,1),(2,2)\}$

d. $\rho = \{(0,0),(1,1),(2,2),(4,4),(6,6),(4,6),(6,4)\}$

Partial Orderings

- A **partial ordering** on a set S is
  - A binary relation on S that is reflexive, antisymmetric, and transitive.
  - Considered *partial* because there may be some elements from S that are not related.
  - Examples of partial orderings
    - On N, $x \rho y \leftrightarrow x \leq y$
    - On $\wp$(N), $A \rho B \leftrightarrow A \subseteq B$
    - On Z+, $x \rho y \leftrightarrow x$ divides y
    - On $\{0,1\}$, $x \rho y \leftrightarrow x = y^2$

Partial Orderings

- A **partially ordered set** is an ordered pair $(S, \preceq)$, where $\preceq$ is a partial ordering on the set S.
  - Also called a **poset**.
- Notation for an arbitrary, partially ordered set is
  - $(S, \preceq)$
  - Replace $\preceq$ with the appropriate relation, such as "x divides y" or "x is less than or equal to y"

Restriction of Partial Ordering

- Let $(S, \leq)$ be a partially ordered set.
- Let A be a subset of S
  - $\leq$ is a set of ordered pairs of elements of S
  - Some members of $\leq$ may be ordered pairs of elements of A.
- Create a new set that consists only of the ordered pairs from $\leq$ that are made up of elements of A
  - This new set is a partial ordering on A, and is called the **restriction** of $\leq$ to A

Partially Ordered Sets

- Let $(S, \preccurlyeq)$ be a partially ordered set
- In a partial ordering, elements in an ordered pair may or may not be equal.
  - The elements in the ordered pair $(4,4)$ are equal $(4 = 4)$.
  - The elements in the ordered pair $(4,10)$ are not equal $(4 \neq 10)$.
- If $x \preccurlyeq y$, but $x \neq y$
  - x is a **predecessor** of y
  - y is a **successor** of x
  - Written as $x \prec y$
- If $x \prec y$ and there is no z with $x \prec z \prec y$
  - Then x is an **immediate predecessor** of y

Partially Ordered Sets

- Practice 8
  - Consider the relation "x divides y" on $\{1, 2, 3, 6, 12, 18\}$
  - Write the ordered pairs (x, y) of this relation
  - Write all the predecessors of 6
  - Write all the immediate predecessors of 6

Hasse Diagram

- Used to visually represent a partially ordered set $(S, \leq)$, given a finite set S.
  - Each element of S is a node of the diagram
  - If x is an immediate predecessor of y
    - The node for y is placed above the node for x.
    - The 2 nodes are connected by a straight-line segment.
  - $S = \{x, y\}, \leq = \{(x,x), (x,y), (y,y)\}$
  - $S = \{x,y,z\}, \leq = \{(x,x),(y,y),(z,z),(x,y),(x,z)\}$

Hasse Diagram Examples

- S = {1,4,8}
  - $\leq$ = {(1,1),(4,4),(8,8),(1,4),(4,8),(1,8)}
- S = {a, b, c, d}
  - $\leq$ = {(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(b,d),(a,d)}
    - Draw the Hasse diagram for the partially ordered set.

Example 10

- Consider $\wp(\{1,2\})$ under the relation of set inclusion. This is a partially ordered set.
  - We already know that $(\wp(N),\subseteq)$ is a partially ordered set.
  - What are the elements of $\wp(\{1,2\})$?
  - What are the ordered pairs of the binary relation $\subseteq$ on S?
  - Draw the Hasse Diagram for the partially ordered set.

Hasse Diagram

- Practice 9
  - Draw the Hasse Diagram for the relation "x divides y" on $\{1, 2, 3, 6, 12, 18\}$
  - $\leq = \{(1,1), (1,2), (1,3), (1,6), (1,12), (1,18),$
    $(2,2), (2,6), (2,12), (2,18), (3,3), (3,6),$
    $(3,12), (3,18), (6, 6), (6,12), (6,18),$
    $(12,12), (18,18)\}$
  - Are all of the elements of S related?

Hasse Diagram

- Hasse diagram of a partially ordered set conveys all the information about the partially ordered set.
  - The ordered pairs of the partially ordered set can be reconstructed from a Hasse diagram.
  - Example (Figure 4.3 in book)

Total Ordering (Chain)

- Total Ordering
  - A partial ordering in which every element of the set is related to every other element.
  - Also called a chain
  - Example (Figure 4.4)
  - "Less than or equal to" relation on N is an example of a total ordering

Least & Minimal Element

- Let $(S, \leq)$ be a partially ordered set.
  - If there is a $y \in S$ with $y \leq x$ for all $x \in S$
    - y is the **least** element of $(S, \leq)$
    - A least element (if it exists) is unique
  - If there is a $y \in S$ such that there is no $x \in S$ with $x < y$
    - y is a **minimal** element of $(S, \leq)$
- In a Hasse diagram
  - A least element is below all others
  - A minimal element has no others below it

Greatest & Maximal Element

- Let $(S, \leq)$ be a partially ordered set.
  - If there is a $y \in S$ with $x \leq y$ for all $x \in S$
    - y is the **greatest** element of $(S, \leq)$
    - A greatest element (if it exists) is unique
  - If there is a $y \in S$ such that there is no $x \in S$ with $y < x$
    - y is a **maximal** element of $(S, \leq)$
- In a Hasse diagram
  - A greatest element is above all others
  - A maximal element has no others above it

Example 11

- Consider the partially ordered set of Practice 9.
  - Identify the least, minimal, greatest, and maximal elements, if they exist.
- Practice 11
  - Draw the Hasse diagram for a partially ordered set with 4 elements in which there are
    - 2 minimal elements but no least element
    - 2 maximal elements but no greatest element
    - Each element is related to exactly 2 other elements

Equivalence Relations

- Equivalence Relation on S
  - A binary relation on a set S that is reflexive, symmetric and transitive
  - An equivalence relation differs from a partial ordering in symmetry.
    - A partial ordering is antisymmetric; an equivalence relation is symmetric
  - Examples
    - On any set S, $x \rho y \leftrightarrow x = y$.
    - On $\{a, b, c\}$, $\rho = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}$
    - $\{x \mid x \text{ is a student in your class}\}$ and ρ is the relation "x sits in the same row as y"

Equivalence Relations

- Partition of a set
  - A collection of nonempty disjoint subsets of S whose union equals S
  - Example
    - $S = \{x \mid x \text{ is a student in your class}\}$
    - $x \rho y \leftrightarrow x$ sits in the same row as y
    - Group all students related to one another
      - Those that sit in row 1, those that sit in row 2, etc.
- Any equivalence relation partitions the set on which it is defined

Equivalence Relations

- Let ρ be an equivalence relation on a set S and $x \in S$
  - $[x]$ denotes the set of all members of S to which x is related (including x itself)
  - $[x]$ is called the **equivalence class** of x.

Equivalence Class

- Example 12
  - $x \ \rho \ y \leftrightarrow$ "x sits in the same row as y"
  - John, Chuck, José, Judy, and Ted all sit in row 3
  - What is [John]?
  - What is [Judy]?

Equivalence Relations & Partitions

- An equivalence relation ρ on a set S determines a partition of S, and
- A partition of a set S determines an equivalence relation on S
- Partial proof in textbook

Practice 14

- For the following equivalence relation, describe the corresponding equivalence classes
  - On {1, 2, 3}, ρ = {(1,1), (2,2), (3,3), (1,2), (2,1)}