CHAPTER 3

Section 3.5

Probability

- Given all possible outcomes of an action (sample space), we want to know the likelihood of a specific event occurring.
  - An event is the occurrence of one of more of the outcomes (a set of outcomes).
- Toss a coin.
  - What is the chance of getting *heads*?
  - What is the chance of getting *tails*?
  - Toss the coin twice, what is the chance of getting *heads* followed by *tails*?

Probability

- We assume outcomes equally likely in the previous examples
  - A balanced coin
    - Getting heads or getting tails is equally likely.
  - An unloaded die
    - Any particular number is just as likely as the others of occurring.

Probability Formula

- Given:
  - Sample Space, *S*, which consists of a finite set of equally likely outcomes
  - Event *E*, $E \subseteq S$
- Probability $P(E)$ of event *E* occurring is

$$P(E) = \frac{|E|}{|S|}$$

Equally likely outcomes

- Roll a die.
  - What is the likelihood (or probability) of rolling a 2?
    - Assuming the die is not loaded, each number is equally likely.
    - How many possible outcomes are there?
      - Our **sample space** is the set {1, 2, 3, 4, 5, 6}
    - Getting a 2 is exactly one of those possibilities.
      - Our **event** is the set {2}.
  - What is the likelihood of getting a 2 or a 3?
    - Same number of outcomes - still talking about 1 roll
    - Our event in this case is the set {2, 3}, which represents 2 of the possible 6 outcomes.

Practice 39

- What is the probability of drawing an ace from a standard (fair) deck of cards?
- Questions to ask
  - What is the sample space, S? What is the size of the sample space, $|S|$?
  - Are the outcomes equally likely?
  - What is the event E? What is the size of the event, $|E|$?

Probability

- Events are sets, so we can use set operations to combine them.
- Given events E_1 and E_2 from S
  - The event $E_1 \cup E_2$ represents the outcomes that are in E_1, E_2, or both.
  - The event $E_1 \cap E_2$ represents the outcomes that are in both E_1 and E_2

Probability

- Think about 2 tosses of a coin
  - What is the probability of getting
    - a heads on **either** toss
    - a heads on **each** toss
    - Let E_1 be the event that you get heads on the first toss
    - Let E_2 be the event that you get heads on the second toss
  - $S = \{HH, HT, TH, TT\}$
    - $E_1 = \{HH, HT\}$
    - $E_2 = \{HH, TH\}$
    - $E_1 \cup E_2 = \{HH, HT, TH\}$
    - $E_1 \cap E_2 = \{HH\}$

Example 65

- Employees participate in a drawing in which one employee name is chosen. There are
  - 5 employees in testing (2 men, 3 women)
  - 23 in development (16 men, 7 women)
  - 14 in marketing (6 men, 8 women)
    - What is the size of the sample space?
    - Let W be the event that a woman's name is drawn
      - What is the size of W? What is the probability of W?
    - Let M be the event that someone from marketing's name was drawn
      - What is the size of M? What is the probability of M?
    - What is the probability that the name of a woman from marketing was drawn?

Table 3.3

	Observation	Justification
1	$0 \leq P(E_1) \leq 1$	$E_1 \subseteq S$, so $0 \leq \lvert E_1 \rvert \leq \lvert S \rvert$
2	The probability of an impossibility is 0.	$E_1 = \varnothing$ so $\lvert E_1 \rvert = 0$
3	The probability of a certainty is 1.	$E_1 = S$ so $\lvert E_1 \rvert = \lvert S \rvert$
4	$P(E_1') = 1 - P(E_1)$	$\lvert E_1' \rvert = \lvert S \rvert - \lvert E_1 \rvert$
5	$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$	See previous discussion.
6	If E_1 and E_2 are disjoint events, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$	Follows from observation 5.

Outcomes not Equally Likely

- Roll an unfair die.
  - 4 is three times more likely to come up than 1, 2, 3, 5, or 6.
  - Can repeat the event that is more likely.
    - S = {1, 2, 3, 4_1, 4_2, 4_3, 5, 6}
    - What is P(3)? P(4)? P(3 or 4)?
  - Can assign a probability distribution to the sample space.

x_i	1	2	3	4	5	6
$p(x_i)$	1/8	1/8	1/8	3/8	1/8	1/8

Probability Distributions

- Consider each distinct outcome in the original sample space as an event and assign it a probability.
  - k different outcomes in S
  - Each outcome x_i is assigned probability $p(x_i)$
    1. $0 \leq p(x_i) \leq 1$
    2. $\sum_{i=1}^{k} p(x_i) = 1$

Probability Distributions

- Consider an event $E \subseteq S$
- $$P(E) = \sum_{x_i \in E} p(x_i)$$
    - Add up all of the probabilities for the individual outcomes in E.

- Practice 41
    - S = {a, b, c}
        - p(a) = 0.2
        - p(b) = 0.3
    - What is p(c)? What is P(a or c)?

Conditional Probability

- Sometimes need to determine the probability of event E_2 given that event E_1 has occurred.
  - The first event occurring affects the number of outcomes for the second event.
  - Practice 42
    - What is the probability of tossing 2 heads given that at least one toss results in heads?

Conditional Probability

- In general, the conditional probability of E_2 given E_1 is

$$P(E_2 \mid E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

- Kind of restricting our sample space to be the outcomes in event E_1

Independent Events

- What if $P(E_2 \mid E_1) = P(E_2)$?
  - In this case, the likelihood of E_2 occurring is not affected by whether or not E_1 occurred.
  - E_1 and E_2 are **independent events**, and we have

$$P(E_2 \mid E_1) = P(E_2) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

  - This can be extended to any finite number of independent events.
  - Can also be used to test whether events are independent.

Independent Events

- Outcome of a specific toss of a coin does not depend on the outcome of any previous tosses.
  - Any toss has a 50/50 chance of getting heads,
    - whether it's the first or the 50th, and
    - whether or not the previous 49 tosses have been heads.

Probability

- "Addition Rule" for Probability
  - If E_1 and E_2 are disjoint events
    - $P(E_1 \cup E_2) = P(E_1) + P(E_2)$
  - Otherwise use principle of inclusion and exclusion.
    - $P(E_1 \cup E_2) = P(E_1) + P(E_2) - \mathrm{P}(E_1 \cap E_2)$
- "Multiplication Rule" for Probability
  - If E_1 and E_2 are independent events
    - $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$
- These can be extended to any finite set of events.

Expected Value

- Student takes 3 tests
  - Set of grades $S = \{g_1, g_2, g_3\}$
  - What is the average test grade?
  - Assuming 3 tests are equally weighted

$$A(g) = \frac{g_1 + g_2 + g_3}{3}$$

$$= \frac{1}{3}(g_1 + g_2 + g_3)$$

$$= g_1\left(\frac{1}{3}\right) + g_2\left(\frac{1}{3}\right) + g_3\left(\frac{1}{3}\right)$$

Expected Value

- If last test counts twice as much, the weighted average grade is

$$A(g) = g_1\left(\frac{1}{4}\right) + g_2\left(\frac{1}{4}\right) + g_3\left(\frac{2}{4}\right)$$

- Assign a probability distribution to S

x_i	g_1	g_2	g_3
$p(x_i)$	1/4	1/4	2/4

- Then

$$A(g) = \sum_{i=1}^{3} g_i p(g_i)$$

Expected Value

- Values in sample space may not be numeric.
  - Can assign a function $X: S \rightarrow \mathbb{R}$
  - The function X is called a **random variable**
    - The function associates a numeric value with each outcome in the sample space
  - Expected value, or weighted average, of the random variable is

$$E(x) = \sum_{i=1}^{n} X(x_i)p(x_i)$$

Example 71

- Toss a fair coin three times.
  - S = {HHH,HHT,HTH,HTT,THH,THT,TTH,TTT}
  - Let random variable X for an outcome be the number of heads in that outcome.
    - Integer value between 0 and 3
    - What is the expected value of X, that is, the expected number of heads in three tosses?

Algorithm Analysis

- Expected value can be used in algorithm analysis to perform average case analysis
  - Expected average amount of work done by an algorithm

$$E(X) = \sum_{i=1}^{n} X(x_i)p(x_i)$$

  - Where $X(x_i)$ is the work done for case i, and
  - $p(x_i)$ is the probability of x_i

Problems 6-12

- A pair of fair dice are rolled
  - What is the size of the sample space?
  - What is the probability of getting
    - snake eyes (two 1s)?
    - doubles (the same number on each die)?
    - a total of 7 on the two dice?
    - a 1 on at least one die?
    - a total on the two dice greater than 10?
    - a total on the two dice that is an odd number?

5-card poker hand

- Problems 29-33

♣	2	3	4	5	6	7	8	9	10	J	Q	K	A
♦	2	3	4	5	6	7	8	9	10	J	Q	K	A
♥	2	3	4	5	6	7	8	9	10	J	Q	K	A
♠	2	3	4	5	6	7	8	9	10	J	Q	K	A

Problems 53-59

- A family has 3 children; boys and girls are equally likely offspring.
  - What is the probability
    - that the oldest child is a boy?
    - of 2 boys and 1 girl?
    - of no girls?
    - of at least one girl?
    - of 3 girls?
    - of 3 girls given that the first two are girls?
    - of at least 1 boy and at least one girl, given that there is at least 1 boy?