CHAPTER 3

Section 3.1

Section 3.1: Sets

- A set is a collection of objects.
  - Usually share some common property P
  - An object with this common property is a member of the set.
  - An object without this common property is not a member of the set.
- What are some examples of sets?
  - Set of students taking CS214-01
  - Set of all positive even integers

Set Notation

- Capital letters denote sets, A, B, C, etc.
- $\in$ denotes membership in a set
  - $x \in C$ means the object *x* is a member, or element, of the set C
- $\notin$ denotes lack of membership in a set
  - $b \notin$ C means the object b is not a member of the set C
- Braces are used to indicate a set
  - A = {violet, chartreuse, burnt umber}
  - violet $\in A$, magenta $\notin A$

Set Notation

- Order is not important in a set.
  - The set {1, 3, 5} is the same as {3, 1, 5}.
- Each element of the set is listed only once.
  - {1, 3, 3, 5} is not correct form.
- Two sets are equal if and only if they contain the same members.
  - $\{a, b, c\} = \{c, b, a\}$
  - $\{a, b\} \neq \{a, b, c\}$
  - $A = B$ means $(\forall x)[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$

Set Notation

- Identifying elements in a set
  - **Finite set**: one with n elements for some nonnegative integer n
    - List the elements.
    - $R = \{\text{red, orange, yellow, green, blue, violet}\}$
- **Infinite set:** infinite number of elements
  - List the elements
    - S = {2, 4, 6, 8, 10, 12, ...}
  - Describe the characterizing property of set S
    - $S = \{x \mid x \text{ is a positive even integer}\}$
    - "The set of all x such that x is a positive even integer."

Set Notation

- In general, notation for a set having the characterizing property P is $\{x \mid P(x)\}$
  - Every member of the set must have the property P.
  - Everything that has the property P must be a member of the set.
- Practice 1: List the elements of the following sets.

  a. $\{x \mid x \text{ is an integer and } 3 < x \leq 7\}$

  b. $\{x \mid x \text{ is a month with exactly 30 days}\}$

  c. $\{x \mid x \text{ is the capital of the United States}\}$

Set Practice

- Practice 2: Describe each of the following sets by giving a characterizing property.

  a. {1, 4, 9, 16}

  b. {2, 3, 5, 7, 11, 13, 17, …}

Standard Sets

- $\mathbb{N}$ = set of all nonnegative integers ($0 \in \mathbb{N}$)
- $\mathbb{Z}$ = set of all integers
- $\mathbb{Q}$ = set of all rational numbers
- $\mathbb{R}$ = set of all real numbers
- $\mathbb{C}$ = set of all complex numbers
- Empty Set (or null set) is the set with no elements denoted by $\varnothing$ or { }
  - What is $\{\varnothing\}$?

Set Example

- Example 2: List the elements in the following sets

$$A = \{x \mid (\exists y)(y \in \{0, 1, 2\} \text{ and } x = y^3)\}$$

$$B = \{x \mid x \in \mathbb{N} \text{ and } (\exists y)(y \in \mathbb{N} \text{ and } x \leq y)\}$$

$$C = \{x \mid x \in \mathbb{N} \text{ and } (\forall y)(y \in \mathbb{N} \rightarrow x \leq y)\}$$

Set Practice

- Practice 3: Describe each set

  a. $A = \{x \mid x \in \mathbb{N}$ and $(\forall y)(y \in \{2, 3, 4, 5\} \rightarrow x \geq y\}$

  b. $B = \{x \mid (\exists y)(\exists z)(y \in \{1, 2\}$ and $z \in \{2, 3\}$ and $x = y+z)\}$

Subsets

- A $\subseteq$ B means A is a subset of B
  - Every member of A is also a member of B.
  - Example:
    - $A = \{2, 3, 5, 12\}$ and $B = \{2, 3, 4, 5, 9, 12\}$
  - Formally, A is a subset of B if
    - $(\forall x)(x \in A \rightarrow x \in B)$
- A $\subset$ B means A is a proper subset of B
    - $A \subseteq B$ and $A \neq B$
    - In the example above, is A a proper subset of B?

Subset Practice

- Practice 6: Let
  - $A = \{x \mid x \in \mathbb{N} \text{ and } x \geq 5\}$
  - $B = \{10, 12, 16, 20\}$
  - $C = \{x \mid (\exists y)(y \in \mathbb{N} \text{ and } x = 2y\}$
- Which of the following statements are true?

$B \subseteq C$

$B \subset A$

$A \subseteq C$

$26 \in C$

$\{11, 12, 13\} \subseteq A$

$\{11, 12, 13\} \subset C$

$\{12\} \in B$

$\{12\} \subseteq B$

$\{x \mid x \in \mathbb{N} \text{ and } x < 20\} \text{ not} \subseteq B$

$5 \subseteq A$

$\{\varnothing\} \subseteq B$

$\varnothing \in A$

$\varnothing \subseteq A$

Practice

- Can also have things like
  - $A = \{a, \{b\}, \{a, b\}, \{ \{a\} \}, b\} \}$
  - How many elements does A have?
    - This is called the **cardinality** of A.
  - Is $\{a\} \in A$?

Set of Sets

- Power set of a set S is a new set - $\wp(S)$ - whose elements are all of the subsets of S
  - $S = \{0, 1\}$
    - $\wp(S) = \{ \varnothing, \{0\}, \{1\}, \{0,1\} \}$
  - Note that the members of a power set are sets.
  - How many members are in the power set for a set with n members?

Problem #12

- Let
  - $R = \{1, 3, \pi, 4.1, 9, 10\}$
  - $S = \{ \{1\}, 3, 9, 10\}$
  - $T = \{1, 3, \pi\}$
  - $U = \{ \{1, 3, \pi\}, 1\}$
- Which of the following are true?

a. $S \subseteq R$
b. $1 \in R$
c. $1 \in S$
d. $1 \subseteq U$
e. $\{1\} \subseteq T$
f. $\{1\} \subseteq S$
g. $T \subset R$
h. $\{1\} \in S$
i. $\emptyset \subseteq S$
j. $T \subseteq U$
k. $T \in U$
l. $T \notin R$
m. $T \subseteq R$
n. $S \subseteq \{1, 3, 9, 10\}$

Binary Operations

- An operation ($\circ$) is a binary operation on a set S if for every *ordered* pair (x, y) of elements of S
  - $x \circ y$ exists and is unique (i.e., the operation is well-defined), and
  - $x \circ y$ is a member of S (i.e., S is **closed** under the operation).
- An ordered pair of numbers is of the form (u, v)
  - Note that (1, 3) is not equal to (3, 1)
  - Two ordered pairs (u, v) and (x, y) are equal only when $u=x$ and $v=y$

Binary Operations

- The symbol $\circ$ is a placeholder for an operation.
  - It should be replaced by the appropriate operation, e.g, $+$
- Example 7: Addition, subtraction and multiplication are all binary operations on $\mathbb{Z}$.
  - Think about addition.
    - Add any 2 numbers from $\mathbb{Z}$. The result is a number in $\mathbb{Z}$.
    - Same for subtraction and addition.
  - Why is division not a binary operation on $\mathbb{Z}$?
- Is addition a binary operation on $\mathbb{N}$?
  - Multiplication? Subtraction?

Binary Operations

- A candidate $\circ$ for an operation can fail to be a binary operation on a set S in any of 3 ways:
  1. There are elements $x, y \in S$ for which $x \circ y$ does not exist.
  2. There are elements $x, y \in S$ for which $x \circ y$ gives more than one result.
  3. There are elements $x, y \in S$ for which $x \circ y$ does not belong to S.

Unary Operations

- For $\#$ to be a unary operation on a set S, it must be true that for any $x \in S$,
  - $x^{\#}$ is well-defined, and
  - S is closed under #.
- Example 12: Let $x^{\#}$ be defined by $x^{\#} = -x$, so that $x^{\#}$ is the negative of x.
  - Is # a unary operation on $\mathbb{Z}$?
  - Is # a unary operation on $\mathbb{N}$?

Operations

- Practice 12
  - Which of the following are neither binary nor unary operations on the given set? Why not?

    a. $x \circ y = x \div y$; S = set of all positive integers.

    b. $x \circ y = x \div y$; S = set of all positive rational numbers.

    c. $x \circ y = x^y$; $S = \mathbb{R}$

    d. $x \circ y$ = maximum of x and y; $S = \mathbb{N}$

Binary Operations

- Given a finite set $S = \{x_1, x_2, \ldots, x_n\}$
  - A binary operation $\circ$ on S can be defined by an array, or table, where element i, j (ith row and jth column) denotes $x_i \circ x_j$.
- Example 14: Let $S = \{2, 5, 9\}$ and let $\circ$ be defined by the array
  - 2 ○ 5 = 2
  - 9 ○ 2 = 5
  - Is ○ a binary operation on S?

○	2	5	9
2	2	2	9
5	5	9	2
9	5	5	9

Operations on Sets

- We can define operations that operate on sets themselves, not just the members of sets
- Given an arbitrary set S, we can define some binary and unary operations on the set $\wp(S)$.
  - S in this case is called the universal set, or the universe of discourse
    - The universal set is the set of all objects under consideration.
    - We want to use subsets from this universal set and perform operations on those subsets.
  - If $S = \mathbb{Z}$, for example, then all subsets will contain only integers.

Operations on Sets

- A binary operation on $\wp(S)$ must act on any two subsets of S to produce a unique subset of S
  - Union operation
    - Let $A, B \in \wp$(S). The union of A and B, denoted by $A \cup B$, is $\{x \mid x \in A \text{ or } x \in B\}$
    - $S = \mathbb{N}$; $A = \{1, 2\}$; $B = \{2, 3\}$
    - $A \cup B = \{1, 2, 3\}$
  - Intersection operation
    - Let $A, B \in \wp$(S). The intersection of A and B, denoted by $A \cap B$, is $\{x \mid x \in A \text{ and } x \in B\}$
    - S, A, and B as defined above
    - $A \cap B = \{2\}$

Operations on Sets

- Example
  - S = the set of all students at UAH
    - Power set of S is various subsets of S, or various groups of students
  - A = the set of students taking CS214
  - B = the set of students taking CS221

  - Union of A and B
    - $A \cup B$ = the set of students taking either CS214 or CS221

  - Intersection of A and B
    - $A \cap B$ = the set of students taking both CS214 and CS221

More Set Operations

- Complement
  - For a set A in $\wp(S)$, the complement of A, A', is $\{x \mid x \in S \text{ and } x \notin A\}$
- Set difference
  - For sets A and B in $\wp(S)$, the set difference:
    $A - B = \{x \mid x \in A \text{ and } x \notin B\}$
    $A - B = \{x \mid x \in A \text{ and } x \in B'\}$
    $A - B = A \cap B'$

More Set Operations

- Disjoint sets
  - Two sets A and B such that $A \cap B = \varnothing$ are said to be disjoint.
  - $A - B$ and $B - A$ are disjoint sets.
    - $(A - B) \cap (B - A) = \varnothing$

Practice 16

- Let

  $A = \{1, 2, 3, 5, 10\}$

  $B = \{2, 4, 7, 8, 9\}$

  $C = \{5, 8, 10\}$

  be subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
- Find
  - $A \cup B$
  - $A - C$
  - $B' \cap (A \cup C)$

Cartesian Product

- Let A and B be subsets of S.
  - The Cartesian product, or cross product, of A and B, denoted by $A \times B$, is defined by

    $$A \times B = \{ (x,y) \mid x \in \text{A and } y \in \text{B} \}$$
  - $A \times A = A^2$
  - A^n is the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of elements of A
- Practice 17
  - Let $A = \{1, 2\}$ and $B = \{3, 4\}$
  - Find $A \times B$, $B \times A$, A^2 and A^3.

Basic Set Identities

Commutative Properties	1a. $A \cup B = B \cup A$ 1b. $A \cap B = B \cap A$
Associative Properties	2a. $(A \cup B) \cup C = A \cup (B \cup C)$ 2b. $(A \cap B) \cap C = A \cap (B \cap C)$
Distributive Properties	3a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ 3b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Properties	4a. $A \cup \varnothing = A$ 4b. $A \cap S = A$
Complement Properties	5a. $A \cup A' = S$ 5b. $A \cap A' = \varnothing$

Countable and Uncountable

- Finite sets
  - Elements in the set are
    - $s_1, s_2, s_3, s_4, \ldots, s_k$
    - Where k is the cardinality of the set (number of elements)
- Infinite set
  - May be able to choose a first element, then a second, then a third, etc.
    - $s_1, s_2, s_3, s_4, \ldots$
    - Eventually, we can get to any element in the set
  - This type of infinite set is **denumerable**.

Countable and Uncountable

- Finite sets and denumerable infinite sets are **countable**.
- Example 21: The set $\mathbb{N}$ is denumerable.
  - Show there is a counting scheme.
    - 0, 1, 2, 3, 4, ...
- Practice 20: The set of even positive integers is denumerable.
- Some infinite sets are uncountable
  - The set of all real numbers between 0 and1