CHAPTER 2

Section 2.2

Principle of Induction

- Climbing an infinitely high ladder – can you reach an arbitrarily high rung?
- 2 Assertions
  1. You can reach the first rung.
  2. Once you get to some rung, you can climb up to the next one.
     - Reached rung $n \rightarrow$ can climb to rung $n+1$
- Given these two assertions are true, can you climb as high as you want?
- What if only one of the assertions is true?

Principle of Induction

- Consider some property of an arbitrary, positive integer
  - P(*n*) means the positive integer *n* has property *P*
  - Goal: Prove that for all positive integers *n*, we have P(*n)*
- Assertions (just like the ladder climbing)
  1. P(1) (1 has property P.)
  2. For any positive integer k, P(*k*) $\rightarrow$ P(*k*+*1*)
     (If any number has property *P*, so does the next number.)
- Prove assertions are true to prove that *P*(*n*) holds for any positive integer *n*.

Principle of Induction

- First Principle of Mathematical Induction

  1. P(1) is true
  2. $(\forall k)[P(k) \text{ true} \rightarrow P(k+1) \text{ true}]$

  $1 \,\&\, 2 \rightarrow P(n)$ true for all positive integers n

- Whenever you want to prove something is true for all $n \geq$ some value, think induction

Proof by Induction

- Step 1: prove the base case, P(1) is true
  - Usually very easy
  - Called the basis, or **basis step**

- Step 2: prove $P(k) \rightarrow P(k+1)$ is true
  - Assume $P(k)$ is true and prove $P(k+1)$ follows from $P(k)$
  - Called the **inductive step**
  - $P(k)$ is called the inductive hypothesis

Proof by Induction

- Example 14: Prove that the equation

  $$P(n): 1 + 3 + 5 + \ldots + (2n-1) = n^2$$

  is true for any positive integer n.

Induction Summary

- Steps in Proof by Induction

Step 1	Prove base case.
Step 2	a. Assume P(*k*).
	b. Prove P(*k*+1), given P(*k*)

§ 2.2, Problem 8

- Prove the following statement is true for every positive integer *n*

$$P(n): 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

§ 2.2, Problem 11

- Prove the following statement is true for every positive integer *n*

$$P(n): 1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

§ 2.2, Example 16

- Prove that for any positive integer n, $2^n > n$

  $P(n)$: $2^n > n$

§ 2.2, Example 17

- Prove that for any positive integer n, the number 2^{2n} - 1 is divisible by 3.

$P(n)$: $2^{2n} - 1 = 3m$, where m is an integer

§ 2.2, Problem 26

- Prove the following is true for every positive **odd** integer n.

$$P(n): (-2)^0 + (-2)^1 + (-2)^2 + \cdots + (-2)^n = \frac{1-2^{n+1}}{3}$$

More Induction

Second Principle of Mathematical Induction

1′. P(1) is true
2′. $(\forall k)$[P(r) true for all r, $1 \leq r \leq k \rightarrow$ P(k+1) true]

1′ & 2′ $\rightarrow$ P(n) true for all positive integers n

Differs from the First Principle in 2′.

More Induction

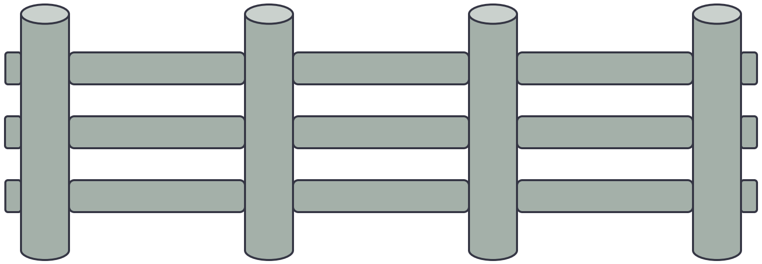
- The two induction principles are equivalent
- Therefore,
  - First principle of induction → second principle of induction
  - Second principle of induction → first principle of induction

Well-Ordering

- Principle of Well-Ordering
  - Every collection of positive integers that contains any members at all has a smallest member.
- Following implications are true (just accept)
  - Second principle of induction $\rightarrow$ first principle of induction
  - First principle of induction $\rightarrow$ well-ordering
  - Well-ordering $\rightarrow$ second principle of induction
- All three principles are equivalent

Example 21

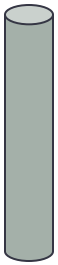
- Prove that a straight fence with n fence posts has n-1 sections for any $n \geq 1$



- For instance, a fence with 4 posts has 3 sections

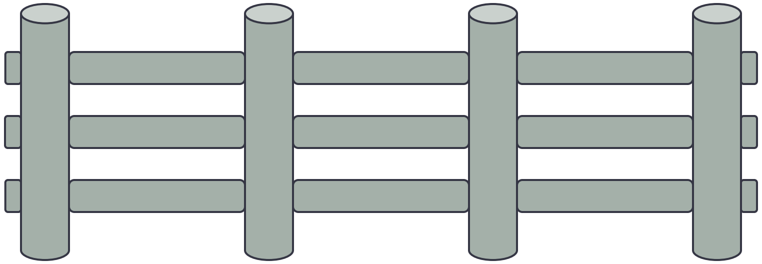
Example 21: First Principle

- P(*n*)*:* a fence with *n* fence posts has *n*-1 sections
- Base Case: P(1)*:* fence with 1 fence post has 0 sections

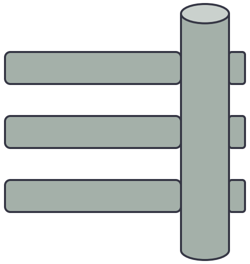


Example 21: First Principle

- Assume *P(k)*
  - A fence with *k* fence posts has *k* - 1 sections
- Prove *P(k+1)*
  - A fence with *k* + 1 fence posts has *k* sections
  - How to relate a fence with k+1 posts to one with k posts so we can use P(k)?



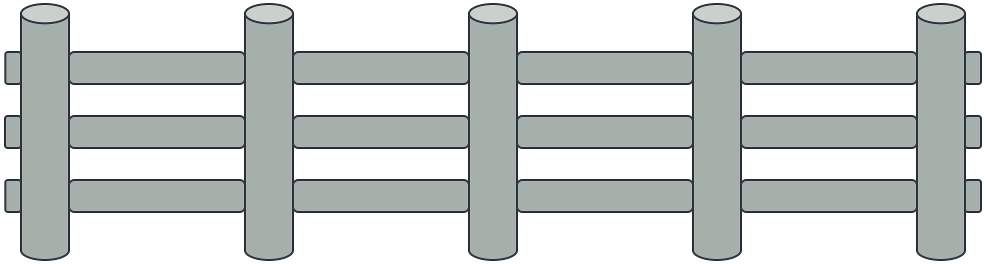
k fence posts



k+1 posts

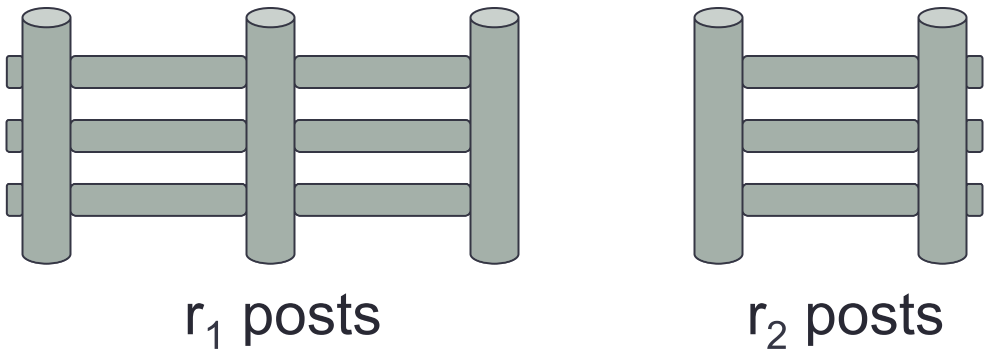
Example 21: Second Principle

- Basis step the same as before
- Assume *P(r)*
  - For all r, $1 \leq r \leq k$, a fence with r fence posts has $r - 1$ sections
- Prove *P(k+1)*
  - A fence with $k + 1$ fence posts has k sections



Example 21: Second Principle

- Split the fence into 2 parts by removing a section



- By the inductive hypothesis, the two parts have (r_1-1) and (r_2-1) sections, so the original fence has

$$(r_1-1) + (r_2-1) + 1 \text{ sections}$$

Example 24

- Prove that any amount of postage greater than or equal to 8 cents can be built using only 3-cent and 5-cent stamps
  - P(n): only 3-cent and 5-cent stamps are needed to build n cents worth of postage
    - Prove P(n) for all $n \geq 8$
  - Base Case: P(8): 8 = 3 + 5
  - Also want to establish 2 additional cases:
    - P(9): 9 = 3 + 3 + 3
    - P(10): 10 = 5 + 5

Example 24

- Assume P(r) for any r, $8 \le r \le k$
- Prove P(k+1)
  - We have already proved P(8), P(9), P(10), so k+1 is at least 11

    $k+1 \ge 11 \Rightarrow (k+1) - 3 = k - 2 \ge 8$
  - By the inductive hypothesis, P(k-2) is true
  - Therefore, k - 2 can be written as a sum of 3s and 5s
  - Adding an additional 3 gives us k+1 as a sum of 3s and 5s.
  - This verifies that P(k+1) is true.

Second Principle

- So, when do you want to use the second principle instead of the first?
  - If you need to go back farther than P(k)
    - Like Example 24
  - If your problem can more easily be split in the middle rather than growing from the end.
    - Like Example 21