CHAPTER 2
SECTION 1

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Proof Techniques

- Moving away from formal arguments of Chapter 1
  - Is the argument universally true?
  - Focused on the relationships between statements
  - Unconcerned with meaning of the statements
- Learn how to prove "real-world" arguments
  - Is the argument true in a specific context?
    - The context is a particular subject, e.g., computer algorithms, graph theory, compiler theory, etc.
    - Adds meaning back into the problems
  - If we prove $P \rightarrow Q$ in the context, then $P \rightarrow Q$ is theorem in that subject.

Proofs and Reasoning

- Two types of reasoning
  - Inductive reasoning
    - Drawing a conclusion based on experience
    - Going from specific examples to generalizations
  - Deductive reasoning
    - A process by which you try to verify the truth or falsity of your conjecture.
    - Goal: To produce a proof of $P \rightarrow Q$, thus making it a theorem, or to find a counterexample that disproves $P \rightarrow Q$.

  - Would you want to only use inductive reasoning in your research?

Counterexample

- When presented with a proof, do you
  - try to prove it is true?
  - try to prove it is false?

- Example 1
  - For a positive integer n, n factorial is defined as $n(n-1)(n-2)...1$ and is denoted by $n!$. Prove or disprove the conjecture "For every positive integer n, $n! \leq n^2$."

Counterexample

- Practice 1: Provide counterexamples to the following conjectures:
  - Every geometric figure with 4 right angles is a square.
  - Every integer less than 10 is bigger than 5.
- What if selecting a counterexample is not an easy task?

Proof Techniques

Proof techniques discussed in Section 2.1

- Exhaustive Proof
- Direct Proof
- Proof by Contraposition
- Proof by Contradiction

Exhaustive Proof

- Examine all possible cases to see if an argument is valid.

- Example 2: prove the conjecture "If an integer between 1 and 20 is divisible by 6, then it is also divisible by 3."

Exhaustive Proof

- Practice 2
  - Prove the conjecture “For any positive integer less than or equal to 5, the square of the integer is less than or equal to the sum of 10 plus 5 times the integer.”

- Exhaustive proof may not be practical in many cases.

Direct Proof

- $P \rightarrow Q$: Assume the hypothesis P and deduce the conclusion Q.
- Formal proof requires a proof sequence leading from P to Q, much like we did for propositional logic.
  - Example 4 in the book illustrates a formal, direct proof.
- For our purposes, an informal proof is sufficient.

Direct Proof

- Informal proof
  - Example 5: Prove that the product of 2 even integers is even.

- What if you cannot easily prove $P \rightarrow Q$?

Proof by Contraposition

- If you can prove $Q' \rightarrow P'$, then you can conclude $P \rightarrow Q$ by making use of the fact that the two are equivalent.
  - Contraposition rule from Chapter 1.

- Example 6: Prove that if the square of an integer is odd, then the integer must be odd.

Proof by Contraposition

- Write the contrapositive of the following statement:
  - If the rain continues, then the river will flood.

- Converse
  - $B \rightarrow A$ is the converse of $A \rightarrow B$.
  - Write the converse of the above statement.

Proof by Contradiction

- Assume $P \to Q$ is false, and try to arrive at a contradiction.
  - How do we make $P \to Q$ false?
  - If you can show a contradiction results from P being true and Q false, then $P \to Q$ must be true.

Proof by Contradiction

- Example 10: Prove the conjecture, "If a number added to itself gives itself, then the number is 0."

- When would proof by contradiction typically be most useful?

Serendipity

- Making a clever observation about a problem.
- Example 12: A tennis tournament has 342 players. A single match involves 2 players. The winner of a match plays the winner of a match in the next round, while losers are eliminated from the tournament. The 2 players who have won all previous rounds play in the final game and the winner wins the tournament. Prove that the total number of matches to be played is 341.

Serendipity

- This “technique” can be useful in computer algorithm analysis.
- Example: How many comparisons are required to find the maximum element in an unsorted list of *n* elements.

Table 2.2 from Book

Proof Technique	**Approach to Prove $P \to Q$**	**Remarks**
Exhaustive Proof	Demonstrate $P \to Q$ for all possible cases	May be used only to prove a finite number of cases
Direct Proof	Assume P, deduce Q	The standard approach - usually the thing to try
Proof by contraposition	Assume Q', deduce P'	Use this if Q' as a hypothesis seems to give more ammunition than P would
Proof by contradiction	Assume $P \wedge Q'$; deduce a contradiction	Use this when Q says something is not true

Useful Definitions from Book

- **Perfect square**: An integer such that $n=k^2$ for some integer k
- **Prime number:** An integer $n > 1$ such that n is not divisible by any integers other than 1 and n
- **Composite number:** A nonprime integer n; that is, n=ab where a and b are integers with $1 < a < n$ and $1 < b < n$
- **x < y**: For 2 numbers x and y, $x < y$ means $y - x > 0$
- **n | m**: n divides m; For 2 integers n and m, n | m means that m is divisible by n, that is $m = k(n)$ for some integer k
- **Absolute value:** Absolute value of a number x, $|x|$, is x if $x \geq 0$ and is $-x$ if $x < 0$